On the recent paper on quark confinement by Tomboulis

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Abstract

We point out missing links in the recent paper by Tomboulis in which he claims a rigorous proof of quark confinement in 4D lattice gauge theory. We also discuss if it is possible to correct his proof.

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1 Introduction

Many physicists have been fascinated by the problem of quark confinement [1] that remains as one of the most mysterious problems in modern physics since

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the last century. It is still an open problem if lattice gauge theory based on a non-abelian gauge group confines quarks for all values of the (bare) coupling constant, and if a continuum limit exists in which quark confinement and asymptotic freedom coexist (this has been questioned by A. Patrascioiu and one of the present authors, see [2] and references therein).

Recently E. T. Tomboulis published a paper in the arXiv [3] in which he claims to present a rigorous proof of quark confinement in 4D lattice gauge theory. We think that though the idea in [3] may be interesting, the proof in [3] depends on an assumption or a claim which remains to be proved. We also discuss if it is possible to correct his proof.

2 Tomboulis's alleged theorem

We summarize his notation and arguments with some simplifications:

1. $\Lambda \subset Z^4$ is the periodic box in Z^4 of size $L_1 \times \cdots \times L_4$ with center at the origin, where $L_i = ab^{n_i}$, $n_i >> 1$, $i = 1, \cdots, 4$ and a, b are positive integers larger than 1.

2. $\Lambda^{(k)} \subset Z^4$ is the periodic box in Z^4 of each side length $L_i b^{-k}$ obtained from Λ by k steps of the renormalization transformation of Migdal-Kadanoff type ($\Lambda = \Lambda^{(0)}$).

3. $f(\{c_j\}, U) = 1 + \sum_{j\neq 0} c_j d_j \chi_j(U)$ where d_j is the dimension of the representation χ_j, and we assume $U \in G = SU(2)$. Moreover $U = U_p = \prod_{b\in\partial p} U_b$ for a plaquette $p \subset \Lambda$ which consists of four oriented bonds $b \in \partial p$

We start with

$$\exp\left[\frac{\beta}{2}\chi_{1/2}(U)\right] = F_0(0) f(\{c_j\}, U) \tag{2.1a}$$

$$f(\{c_j\}, U) = 1 + \sum_{j\neq 0} c_j d_j \chi_j(U) \tag{2.1b}$$

$$F_0(0) = \int \exp\left[\frac{\beta}{2}\chi_{1/2}(U)\right] dU \tag{2.1c}$$

where dU is the Haar measure of $G = SU(2)$. We then apply the renormalization group (RG) recursion formulas of Migdal-Kadanoff type [4] with

correction parameters. If there are no correction parameters (the standard recursion formula), we have

$$\begin{aligned}
f^{(n-1)}(U) &= f(\{c_j(n-1)\}, U) \\
&= 1 + \sum_{j\neq 0} c_j(n-1) d_j \chi_j(U) \\
&\to f^{(n)}(U) = f(\{c_j(n)\}, U)
\end{aligned}$$

where

$$\begin{aligned}
f^{(n)}(U) &= \frac{1}{F_0(n)} \int \left(f^{(n-1)}(UU_1) f^{(n-1)}(U_1^{-1}U_2) \cdots f^{(n-1)}(U_{b^2})\right)^{b^2} \prod dU_k \\
F_0(n) &= \left(\int [f^{(n-1)}(U)]^{b^2} dU\right)^{b^2}
\end{aligned}$$

or equivalently

$$c_j(n) = \frac{F_j(n)}{F_0(n)} \tag{2.2}$$

$$F_j(n) = \left(\int [f^{(n-1)}(U)]^{b^2} \frac{\chi_j(U)}{d_j} dU\right)^{b^2} \tag{2.3}$$

in terms of the coefficients of the character expansions. Then [5]:

Theorem 2.1 *For $D \leq 4$ and for $G = SU(N)$ or $G = U(N)$,*

$$\lim_{n\to\infty} c_j(n) = 0 \qquad \text{for } j \neq 0$$

These recursions are just approximative and yield upper bounds for the partition functions [3]. Then two parameters $\alpha \in (0,1]$ and $t > 0$ and a function $h(\alpha, t) \in (0,1]$ are introduced in [3] so that this transformation is numerically exact:

$$Z = \int dU_\Lambda \prod_{p\subset\Lambda} f(\{c_j\}, U_p) \tag{2.4a}$$

$$= [F_0(1)^{h(\alpha,t)}]^{|\Lambda^{(1)}|} \int dU_{\Lambda^{(1)}} \prod_{p\subset\Lambda^{(1)}} f(\{\tilde{c}_j(\alpha)\}, U_p) \tag{2.4b}$$

$$= \tilde{Z}_1(\tilde{c}(\alpha), t) \tag{2.4c}$$

where $dU_\Lambda = \prod_{b\in\Lambda} dU_b$,

$$\begin{aligned}\tilde{Z}_1(\tilde{c}(\alpha),t) &= [F_0(1)^{h(\alpha,t)}]^{|\Lambda^{(1)}|} Z_1 &\quad (2.5a)\\ Z_1 &= \int dU_{\Lambda^{(1)}} \prod_{p\subset\Lambda^{(1)}} f(\{\tilde{c}_j(\alpha)\}, U_p) &\quad (2.5b)\end{aligned}$$

and as usual, $U_p = \prod_{b\in\partial p} U_b$ are plaquette actions defined as the product of group elements $U_b = U_{(x,x+e_\mu)} \in G$ attached to the (oriented) bonds $b \in \Lambda$. Moreover

$$\begin{aligned}h(\alpha,t) &= \exp[-t(1-\alpha)/\alpha] &\quad (2.6a)\\ \tilde{c}_j(\alpha) &= \tilde{c}_j^{(1)}(\alpha) = \alpha c_j(1) &\quad (2.6b)\\ c_j(1) &= \frac{F_j(1)}{F_0(1)} &\quad (2.6c)\end{aligned}$$

and $\alpha = \alpha(t) \in [0,1]$ is chosen so that the above relation becomes exact.

Tomboulis also introduces a vortex $V = \{v \subset \Lambda\}$ [6] which is a collection of plaquettes $\{v = \{x_0+ne_3, x_0+e_1+ne_3, x_0+e_2+ne_3, x_0+e_1+e_2+ne_3\}, n = 0,1,,\cdots,L_3\}$, namely a plaquette $p = (x_0, x_0+e_1, x_0+e_1+e_2, x_0+e_2)$ in an $x_1 - x_2$ plane and its translations along one of the axis normal to p (say, 3rd or 4th axis). We define

$$Z^- = \int dU_\Lambda \prod_{p\subset\Lambda} f(\{c_j\}, (-1)^{\nu(p)} U_p) \qquad (2.7)$$

where

$$\nu(p) = \begin{cases} 0 & \text{if } p \notin V \\ 1 & \text{if } p \in V \end{cases} \qquad (2.8)$$

and then

$$\begin{aligned}Z^- = &\int dU_\Lambda \prod_{p\subset\Lambda\backslash V} (1 + \sum_{j\neq 0} c_j d_j \chi_j(U_p))\\ &\times \prod_{q\subset V} (1 + \sum_{j\neq 0} (-1)^{2j} c_j d_j \chi_j(U_q)) \qquad (2.9)\end{aligned}$$

Note that the position of the vortex $V \subset \Lambda$ can be freely moved in the $x_1 - x_2$ plane by gauge invariance. It is easy to see that

Theorem 2.2 *Assume $c_j \geq 0$. Then*
(1) The measure $d\mu_\Lambda$ defined by

$$d\mu_\Lambda = \prod_p (1 + \sum_{j \neq 0} c_j d_j \chi_j(U_p)) dU_\Lambda$$

is reflection positive with respect to all planes dividing the periodic box Λ.
(2) The measure $d\mu_\Lambda^{(+)}$ defined by

$$d\mu_\Lambda^+ = \left[\prod_p (1 + \sum_{j \neq 0} c_j d_j \chi_j(U_p)) + \prod_p (1 + \sum_{j \neq 0} c_j d_j \chi_j((-1)^{\nu(p)} U_p)) \right] dU_\Lambda$$

is reflection positive with respect to all planes without bisecting $V \subset \Lambda$

Remarks 2.1 *(1) Though the reflection positivity of interactions plays a very important role in general, only $\{c_j(n) \geq 0\}$ is used in the paper [3].*
(2) The vortex structure in Z^- is kept in $Z^-_{\Lambda^{(n)}}$ by the RG transformations of Migdal-Kadanoff type.

The main claim in [3] is:

Claim 2.1 *There exist $t \geq 0$ such that*

$$1 + \frac{Z_\Lambda^-(\{c_j\})}{Z_\Lambda(\{c_j\})} = 1 + \frac{Z^-_{\Lambda^{(1)}}(\{\tilde{c}_j^{(1)}(\alpha(t))\})}{Z_{\Lambda^{(1)}}(\{\tilde{c}_j^{(1)}(\alpha(t))\})}$$

where

$$\tilde{c}_j^{(1)}(\alpha) = \alpha c_j(1).$$

If this were correct, we could have

$$\frac{Z_\Lambda^-(\{c_j\})}{Z_\Lambda(\{c_j\})} = \frac{Z^-_{\Lambda^{(n)}}(\{\tilde{c}_j^{(n)}(\alpha(t))\})}{Z_{\Lambda^{(n)}}(\{\tilde{c}_j^{(n)}(\alpha(t))\})}$$

by induction. Since $\{c_j^{(n)} \geq 0\}$ tends to the high temperature fixed point(i.e. $\{c_j(n)\} \to 0$) as $n \to \infty$ if the dimension is ≤ 4, whether G is abelian or non-abelian (see the remark below and [5]), this would mean strict positivity of 't Hooft's string tension and then establish permanent confinement of quarks in the sense of Wilson at least for all values of the bare coupling constant

[6, 7] in 4 dimensional lattice gauge theory, thereby solving a longstanding problem in modern physics.

This cannot be correct, however, since there exists a deconfining Kosterlitz-Thouless (KT) type transition in 4D lattice gauge theory based on abelian gauge groups. But it is meaningful to ask why this wrong conclusion is reached.

Remarks 2.2 *(1) The introduction of* $0 < \alpha \leq 1$ *into* $1 + \sum_{j\neq 0} c_j d_j \chi_j(U)$ *does not violate conditions (positivity, analyticity, class functions etc.) on* $f^{(n)}(v)$ *in [5] since*

$$\begin{aligned} 1 + \sum_{j\neq 0} \alpha c_j d_j \chi_j(U) &= (1-\alpha) + \alpha \left(1 + \sum_{j\neq 0} c_j d_j \chi_j(U) \right) \\ &= (1-\alpha) + \alpha \, (\textit{solution by the Migdal-Kadanoff formula}) \end{aligned}$$

and $0 < \alpha \leq 1$. *Then* $\{c_j(n) \geq 0\}$ *tends to 0 as* $n \to \infty$.
(2) Tomboulis introduces another interpolation parameter $r \in (0, 1]$ *into the Migdal-Kadanoff formula (*b^2 *convolutions are replaced by* $b^2 r$ *). The parameter* r *increases the dimension* D *from* $D = 4$ *to* $D \geq 4$ *from the point of view of the renormalization groups. So we set* $r = 1$ *in this paper. The introduction of* r *does not change our argument.*
(3) The conjecture raised in [6] is proved rigorously in [7]. Namely 't Hooft's string tension is smaller than or equal to Wilson's string tension.

3 Tomboulis's arguments revisited

We follow arguments in [3]. First, using the fact that the partition function $Z = Z_\Lambda$ increases by the Migdal-Kadanoff recursion formula [3], he introduces two interpolation parameters α (Z increases as $\alpha \nearrow 1$) and t (the factor $[F_0(n)]^{h(\alpha,t)}$ decreases as t increases). Then he claims that there exist functions $\alpha(t)$ and $\alpha^+(t)$ such that

$$Z_\Lambda(\{c_j\}) = [F_0(1)]^{h(\alpha(t),t)|\Lambda^{(1)}|} Z_{\Lambda^{(1)}}(\{\tilde{c}_j(\alpha(t))\}) \tag{3.1a}$$

$$Z_\Lambda^+(\{c_j\}) = [F_0(1)]^{h(\alpha^+(t),t)|\Lambda^{(1)}|} Z_{\Lambda^{(1)}}^+(\{\tilde{c}_j(\alpha^+(t))\}) \tag{3.1b}$$

where

$$Z^+ = \frac{1}{2}(Z + Z^-)$$

and the right hand sides are independent of t.

The author of [3] then claims is that there exists a $t_* > 0$ such that

$$\alpha(t_*) = \alpha^+(t_*) \tag{3.2}$$

which yields

$$\frac{Z^+_\Lambda(\{c_j\})}{Z_\Lambda(\{c_j\})} = \frac{Z^+_{\Lambda^{(1)}}(\{\tilde{c}_j\})}{Z_{\Lambda^{(1)}}(\{\tilde{c}_j\})} \tag{3.3}$$

where

$$\tilde{c}_j = \tilde{c}_j(\alpha(t_*))$$

But this is not proved in the paper [3]. We follow his arguments by introducing the derivatives of the free energies with respect to α:

$$A(\alpha) = \frac{1}{\log(F_0(1))|\Lambda^{(1)}|}\frac{\partial}{\partial\alpha}\log Z_1(\{\tilde{c}_j(\alpha)\}) \tag{3.4a}$$

$$A^+(\alpha) = \frac{1}{\log(F_0(1))|\Lambda^{(1)}|}\frac{\partial}{\partial\alpha}\log Z^+_1(\{\tilde{c}_j(\alpha)\}) \tag{3.4b}$$

(Here and hereafter we write $Z_n = Z_{\Lambda^{(n)}}$.) Then the author of [3] proves that if

$$A(\alpha) > A^+(\alpha) \tag{3.5}$$

then equation (3.2) has the solution. Namely equation (3.2) is reduced to inequality (3.5) which is, as far as we can see, not proven, even though the author remarks at the beginning of page 27 of [3]:

> Assume now that under successive decimations the coefficients $c^U_j(m)$ evolve within the convergence radius $\cdots$. Taking then n sufficiently large, we need establish inequality (5.15) (namely $A \geq A^+$) only at strong coupling.

Furthermore the discussion soon after ineq.(5.6) in [3] is written as if equation (3.2) were trivial or proven, and the following equation ((5.21) in [3]) is written without any explanation:

$$\cdots = \frac{A^{(-)}(\xi)}{A(\xi)} \leq 1 \tag{5.21}$$

It is not clear where and how his claim is proven for large β where the high-temperature expansion never works!

Inequality (3.5) is not trivial at all since it involves derivatives of log's of presumably large functions. (This inequality is obvious when $\{c_j \geq 0\}$ are small and the formal expansion converges.)

The implicit function theorem is used in [3] to prove the existence of $t = t_*$ satisfying (3.2). As in [3], we introduce

$$\Psi(\lambda, t) = h(\alpha(t), t) + \frac{1}{\log(F_0(1))|\Lambda^{(1)}|} \times \Big((1-\lambda)\log Z_1^+(\{\tilde{c}_j(\alpha^+(t))\}) + \lambda \log Z_1^+(\{\tilde{c}_j(\alpha(t))\}) - \log Z^+(\{c_j\})\Big)$$

where

$$\begin{aligned} \log Z^+(\{c_j\}) &= \log\left[F_0(1)^{h(\alpha^+(t),t)|\Lambda^{(1)}|} Z_1^+(\{\tilde{c}(\alpha^+(t))\})\right] \\ &= \log\left[F_0(1)^{h(\alpha^+(t_I),t)|\Lambda^{(1)}|} Z_1^+(\{\tilde{c}(\alpha^+(t_I))\})\right] \end{aligned}$$

by the parametrization invariance (t-invariance) of the partition function (this is the definition of α^+). Then

$$\Psi(\lambda = 0, t) = h(\alpha(t), t) - h(\alpha^+(t_I), t_I) \tag{3.6a}$$

$$\Psi(\lambda = 1, t) = \frac{1}{\log F_0(1)|\Lambda^{(1)}|} \times \left(\log\left[F_0(1)^{h(\alpha(t),t)|\Lambda^{(1)}|} Z_1^+(\{\tilde{c}_j(\alpha(t))\})\right] - \log Z^+(\{c_j\})\right) \tag{3.6b}$$

We can assume that the equation $\Psi(\lambda = 0, t) = 0$ is solved by $t = t_0$, and the equation $\Psi(\lambda, t) = 0$ is our required equation, and we want to know if the solution $t = t(\lambda)$ with $t(0) = t_0$ can be continued to $t(1)$. We have

$$t(\lambda) = t_0 + \int_0^\lambda F(s, t(s))ds \tag{3.7}$$

$$F(s, t(s)) = -\frac{\Psi_s(s, t(s))}{\Psi_t(s, t(s))} \tag{3.8}$$

which can be analytically solved by iteration if $F(s, t)$ is bounded in the

region. Here

$$\Psi_t(\lambda,t) = \left[1-\frac{h_\alpha(\alpha,t)+\lambda A^+(\alpha)}{h_\alpha(\alpha,t)+A(\alpha)}\right] h_t(\alpha,t) \tag{3.9}$$

$$\begin{aligned}\Psi_\lambda(\lambda,t) &= \frac{1}{\log F_0(1)|\Lambda^{(1)}|} \\ &\quad \times \left(\log Z_1^+(\{\tilde{c}_j(\alpha(t))\}) - \log Z^+(\{\tilde{c}_j(\alpha^+(t)\})\right)\end{aligned} \tag{3.10}$$

and

$$h_t(\alpha,t) = -\frac{1-\alpha}{\alpha} h(\alpha,t) \tag{3.11}$$

So if $A^+(\alpha) > A(\alpha)$ $(0<\alpha<1)$, then $\Psi_t(\lambda,t)=0$ for some $0<\lambda<1$ and the integrand $F(s,t)$ diverges at some $s=s_0<1$. Thus we cannot expect that the solution can be continued to yield $t(1)$.

As is pointed out in [3] and as is easily proved, we can prove $A > A^+$ if $\{c_j \geq 0\}$ are small and the high-temperature expansion converges. But we do not see that the proof of his claim for large β is given in the paper [3].

4 Discussion

If the conventional wisdom of quark confinement in 4D non-abelian lattice gauge theory is correct, the alleged theorem in [3] may hold for $G=SU(N)$. But it is again a very subtle problem to show the existence of t such that $\alpha(t)=\alpha^+(t)$ since it does not exist in the case of $G=U(1)$.

Though the Migdal-Kadanoff RG recursion formulas cannot distinguish non-abelian groups from abelian ones, the velocities of the convergences of $\{c_j(n)\}_{j=1/2}^\infty$ to 0 as $n\to\infty$ are very different. We are rather skeptical about the idea that the problem of quark confinement can be solved by soft analysis like this, but *if* the Migdal-Kadanoff RG formulas should play a role in a rigorous proof of quark confinement in lattice gauge theory, this fact would certainly have to come into play.

In the case of $D=3$, in which case $\{c_j(n)\}_{j=1/2}^\infty$ converges to 0 exponentially fast as $n\to\infty$, we may have a chance to apply his idea to the problem of quark confinement in 3D lattice gauge theory which is not yet solved. But so far, we do not know the method.

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